CS 6505 : Computability and Algorithms Spring 2014

Traveling Salesman Problem

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1 Traveling Salesman Problem

- $G = (V, E)$ is a complete undirected graph
- Non-negative integer cost $c(u, v)$ with each edge $(u, v) \in E$.
- **TSP**: Find a Hamiltonian cycle of G with minimum cost.
- **TSP with triangle inequality:** The cost function c satisfies the *triangle inequality* if for all vertices $u, v, w \in V$:
$$c(u, w) \leq c(u, v) + c(v, w).$$
- TSP with triangle inequality is NP-Complete: also known as metric TSP or constrained TSP.
  [Proved in HW]
- The TSP has several applications in planning, logistics, VLSI design, DNA sequencing etc.
- Probably the most well-studied problem in combinatorial optimization.

2 Inapproximability of TSP

Claim: If $P \neq \mathsf{NP}$, then for any polynomial time computable function $\rho(n)$, there is no polynomial time $\rho(n)$-approximation algorithm. for the general TSP.

- Suppose that there is a polynomial time ρ approximation algorithm, say $\mathcal{A}$ for TSP.
- We will show that we can use $\mathcal{A}$ to decide the *Hamiltonian Cycle* problem is which is NP-complete thus showing that $\mathsf{P} = \mathsf{NP}$.
  - Let $G = (V, E)$ be an undirected graph.
  - Construct a complete graph $G' = (V, E')$ from V.
  - For each $u, v \in E'$, assign an integer cost:
    * $c(u, v) = 1$ if $(u, v) \in E$ and
    * $c(u, v) = \rho \times |V| + 1$ if $(u, v) \notin E$.
  - Run $\mathcal{A}$ on G' with this cost function on the edges.
  - Suppose G has a Hamiltonian cycle.
    * The cost of this cycle in G' is $|V|$.
    * $\mathcal{A}$ returns a tour whose cost is at most $\rho \times |V|$.
  - Suppose G has no Hamiltonian cycle.
    * The cost of any Hamiltonian cycle in G' is $> \rho \times |V|$:
      · Any Hamiltonian cycle in G' must include an edge not in E.
      · Any Hamiltonian cycle has cost at least $(\rho \times |V| + 1) + (|V| - 1)$ which is $> \rho \times |V|$.

3 A 2- approximation algorithm for metric TSP

1. Construct a minimum spanning tree (MST) T.
2. Double every edge of T to get an Eulerian graph.
3. Find an Eulerian tour W on this graph. We can take a preorder traversal of T.
4. Let L be the list of vertices obtained by deleting all duplicates in W by keeping, for all vertices u, only the first visit to the vertex u.
5. Let H be the cycle corresponding to this traversal.

4 Analysis

Claim: The algorithm given above is a 2-optimal approximation algorithm.

- Let H^* be an optimal TSP tour.
- Then, $C(T) \leq C(H^*)$.
  - Deleting an edge from H^* gives a spanning tree of G.
- Let W be a list of vertices from a preorder traversal of T before removing duplicates.
- Then, $C(W) = 2C(T)$:
  - Every edge of T is traversed exactly twice in W.
- Therefore, $C(W) \leq 2C(H^*)$.
- Let H be the cycle obtained by deleting all duplicates in W by keeping, for all vertices u, only the first visit to the vertex u.
- Then, $C(L) \leq C(W)$:
  - Let W' be the list obtained from W after the deletion of some vertices.
  - Say a vertex v occurring in the order u, v, w in W' is deleted.
  - Then, the cost of the resulting list is at most the cost of W':
    * There is an edge between u and w since G is complete.
    * By triangle inequality, $c(u, w) \leq c(u, v) + c(v, w)$.
- Exercise: The analysis is tight!

5 Christofides Algorithm: 3/2 approximation for metric TSP

1. Construct a minimum spanning tree T.
2. Compute a minimum cost perfect matching M on the set of odd-degree vertices of T. Add M to obtain an Eulerian graph.
3. Find an Eulerian tour W on this graph.
4. Let L be the list of vertices obtained by deleting all duplicates in W by keeping, for all vertices u, only the first visit to the vertex u.
5. Let H be the cycle corresponding to this traversal.

6 Analysis

- Key idea: Use perfect matching in odd degree vertices of MST to obtain an Eulerian graph in step 2.
- Let $S \subseteq V$ and $|S|$ is even and M is a minimum cost perfect matching on S then $cost(M) \leq \mathsf{Opt}/2$
  - Let H^* be the optimal TSP tour and $cost(H^*) = \mathsf{Opt}$
  - Let H' be the tour on S by short-cutting H^*.
  - By triangle inequality, $cost(H') \leq \mathsf{Opt}$.
  - Now H' is union of two perfect matchings on S.
  - The cheaper of these two matchings has cost $\leq cost(H')/2 \leq \mathsf{Opt}/2$.
- $cost(H) \leq cost(T) + cost(M) \leq \mathsf{Opt} + \mathsf{Opt}/2 \leq 3/2\mathsf{Opt}$.
- The Analysis is tight!
- Exercise: Find such a tight example.

7 Other Comments:

- It is a BIG open question in theoretical computer science to get a $3/2 - \epsilon$ approximation for metric TSP for any $\epsilon > 0$.
- The Euclidean TSP, or planar TSP, is the TSP with the distance being the ordinary Euclidean distance.
- The Euclidean TSP is a particular case of the metric TSP, since distances in a plane obey the triangle inequality.
- Sanjeev Arora and Joseph S. B. Mitchell were awarded the Gödel Prize in 2010 for their concurrent discovery of a PTAS for the Euclidean TSP.
- There are commercial softwares like Concorde which can solve most of the problems with millions of cities within a small fraction of 1% of the optimal.

8 Resources:

I am following chapter 2.4 (The traveling salesman problem) of [1] for the lectures. The book is freely available online: *http://www.designofapproxalgs.com/*. You can also see chapter 3 (Steiner Tree and TSP) from [2].

References

[1] Williamson, David P and Shmoys, David B. *The Design of Approximation Algorithms.* Cambridge University Press 2011.

[2] Vazirani, Vijay V. *Approximation Algorithms*. Springer 2001.